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The Hoàng–Reed Conjecture for $\delta^+ = 3$

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ABSTRACT

The Hoàng–Reed Conjecture states that a digraph with minimum out-degree d contains d dicycles $C_1, C_2, \dots, C_d$ such that C_k intersects $\bigcup_{i<k} C_i$ on at most one vertex, for each k. It was made as a more structural approach to the Caccetta–Häggkvist Conjecture. We introduce the concept of a nearest separator and use it to prove a stronger version of the Hoàng–Reed Conjecture for the $\delta^+ = 2$ case. With these two tools we go on to prove the $\delta^+ = 3$ case.

1. Introduction

Hoàng and Reed [6] made a conjecture to stimulate a structural approach to the famous Caccetta–Häggkvist Conjecture. The Hoàng–Reed Conjecture states that the minimum out-degree $\delta^+(D)$ of a digraph D forces a type of subdigraph called a circuit-forest. A d-circuit-forest is a digraph F which is given by a union of d dicycles such that for any two vertices u and v, there is at most one (u, v)-dipath in F. Equivalently, we can label the d dicycles of F as $C_1, \dots, C_d$ such that C_k intersects $\bigcup_{i<k} C_i$ on at most one vertex for each $k = 2, \dots, d$.

Conjecture 1.1 (*Hoàng and Reed [6]*). *Each digraph D contains a $\delta^+(D)$-circuit-forest.*

The pigeonhole principle implies that one of the dicycles of a δ^+-circuit-forest in a digraph with n vertices has a length of at most $\lceil n/\delta^+ \rceil$. Thus, the Caccetta–Häggkvist Conjecture follows from the Hoàng–Reed Conjecture.

Conjecture 1.2 (*Caccetta and Häggkvist [3]*). *Each digraph on n vertices with minimum out-degree $\delta^+ \geq 1$ contains a dicycle of length at most $\lceil n/\delta^+ \rceil$.*

Caccetta and Häggkvist [3] proved their conjecture for $\delta^+ = 2$ and Hamidoune [4] proved the $\delta^+ = 3$ case. Hoáng and Reed [6] proved it for $\delta^+ \leq 5$, and then made their conjecture. Many other approximate results have been attained and the reader is referred to Sullivan [7] for a survey of topics related to the Caccetta–Häggkvist Conjecture.

Recently, Havet, Thomassé and Yeo [5] proved the Hoàng–Reed Conjecture for tournaments. They observed that their techniques give more insight into tournaments rather than the structure of dicycles in general digraphs. It is possible that the techniques we use in this paper represent a first step towards understanding the Hoàng–Reed Conjecture in general digraphs.

Thomassen [9] provided a proof of the $\delta^+ = 2$ case of the Hoàng–Reed Conjecture by inductively finding two dicycles that intersect on a single vertex. Observe that if the out-neighborhood of a vertex v is δ^+-connected to the in-neighborhood

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of v, then the digraph contains a collection of δ^+ dicycles, any two of which intersect exactly at v. Such a circuit-forest would satisfy the Hoàng–Reed Conjecture, but Thomassen has also established the following theorem which shows that circuit-forests of these types can not necessarily be found.

Here and throughout this paper, if S and T are subsets of $V(D)$, then $\kappa_D(S, T)$ is equal to the maximum number of pairwise disjoint (S, T)-dipaths in D.

Theorem 1.3 (*Thomassen [10]*). *For every integer $k \geq 3$, there is a digraph D with $\min\{\delta^+(D), \delta^-(D)\} \geq k$ such that $\kappa_D(N_D^+(u), N_D^-(u)) \leq 2$ for all vertices $u \in V(D)$.*

This says that, in certain instances, the intersection graph of the dicycles of a δ^+-circuit-forest in D can not necessarily be taken as complete or even to contain a triangle. However, the following strengthening of the Hoàng–Reed Conjecture might be true.

Conjecture 1.4. *Each digraph D contains $d = \delta^+(D)$ dicycles $C_1, \ldots, C_d$ such that $\bigcup_{i=1}^d C_i$ is a d-circuit-forest and the intersection graph of $\{V(C_i)\}_{i=1}^d$ is a forest.*

If this is true, then such a d-circuit-forest contains $\lceil d/2 \rceil$ disjoint dicycles, since the intersection graph is bipartite. This implies the following related conjecture.

Conjecture 1.5 (*Bermond and Thomassen [2], [8]*). *For every positive integer k, every digraph D with $\delta^+(D) \geq 2k - 1$ contains k disjoint dicycles.*

In this paper we prove the following lemma, which is a strengthening of the Hoàng–Reed Conjecture for $\delta^+ = 2$. We then use it to prove the subsequent theorem.

Main Lemma. *Let D be a digraph. Let s and t be two distinct vertices of D such that there is a (v, t)-dipath for all $v \in V(D)$. If $d_D^+(v) \geq 2$ for all $v \in V(D) \setminus \{t\}$, then D contains an (s, t)-dipath P and a dicycle C such that $|V(P) \cap V(C)| = 1$ and $t \notin V(C)$.*

Main Theorem. *If D is a digraph with $\delta^+(D) \geq 3$, then D contains a 3-circuit-forest.*

The concept of a nearest separator is used in proving the results of this paper. Section 2 covers the definition of a nearest separator and a few of the basic properties of nearest separators. Sections 3 and 4 cover the $\delta^+ = 2$ and $\delta^+ = 3$ cases of the Hoàng–Reed Conjecture, respectively.

Notation and terminology

As usual, D will always denote a finite digraph without loops and without multiple arcs, though oppositely oriented arcs (e.g., uv and vu) are allowed. We use $V(D)$ to denote the set of vertices. The set of arcs of D is denoted by $A(D)$. If $B \subset V(D)$, the set of out-arcs of B is $A_D^+(B)$, i.e., the set of arcs which have their tail in B and head in $V(D) \setminus B$. We use $D[B]$ to denote the subdigraph of D induced by B. Thus, $D[B] = D - (V(D) \setminus B)$.

If H is a strongly connected component of D (or *strong component* for short), we call H a *minimal strong component* if $A_D^+(V(H))$ is empty. Equivalently, a minimal strong component sits at the end of some acyclic ordering of the strong components of D. Similarly, H is a *maximal strong component* if $A_D^-(V(H))$ is empty.

If $u \in V(D)$, we use $N_D^+(u)$ to denote the *out-neighborhood* of u, the set of vertices incident with heads of arcs of $A_D^+(u)$. We use $d_D^+(u)$ to denote the *out-degree* of u, or $\left|N_D^+(u)\right|$. The terms $N_D^-(u)$ and $d_D^-(u)$ are defined similarly, but refer to the in-neighborhood of u. The *minimum out-degree* of D is $\delta^+(D) = \min\{d_D^+(u) : u \in V(D)\}$ and the *maximum out-degree* of D is $\Delta^+(D) = \max\{d_D^+(u) : u \in V(D)\}$.

If $P = p_1p_2 \cdots p_a$ and $Q = q_1q_2 \cdots q_b$ are dipaths, we use $p_i|P$ to denote the dipath $p_ip_{i+1} \cdots p_a$. Similarly, $P|p_i$ denotes the dipath $p_1p_2 \cdots p_i$. If there is an arc from p_a to q_1, then we use $P \cdot Q$ to denote the directed walk $p_1p_2 \cdots p_aq_1q_2 \cdots q_b$. The vertex p_a is called the *terminal vertex* of P and the vertex q_1 is called the *initial vertex* of Q.

Let $S, T \subset V(D)$. A (S, T)-*dipath* is a directed path that has its initial vertex in S, its terminal vertex in T and no other vertices in $S \cup T$. The *lower set* of S in D or 'D below S' is $D \downarrow S = \{u \in V(D) : \exists p_1 \cdots p_a u \subset D, p_1 \in S\}$. Thus, D below S is the set of vertices u such that there is a (S, u)-dipath in D. Recall that, by Menger's Theorem, $\kappa_D(S, T) = \min\{|X| : X \subset V(D), T \cap ((D - X) \downarrow S) = \emptyset\}$ is also the maximum number of pairwise disjoint (S, T)-dipaths.

2. Lemmas on nearest separators

Definition 2.1. Let S and T be two non-empty sets of vertices in a digraph D. An (S, T)-separator is a set X of $\kappa_D(S, T)$ vertices such that there is no (S, T)-dipath in $D - X$. We say that X is a nearest (S, T)-separator if the lower set $(D - X) \downarrow S$ is minimal with respect to inclusion among all (S, T)-separators, i.e., if Y is an (S, T)-separator and $(D - Y) \downarrow S \subset (D - X) \downarrow S$, then $(D - Y) \downarrow S = (D - X) \downarrow S$ and $Y = X$.

Our definition of an (S, T)-separator is slightly more restrictive than that given by Bang-Jenson and Gutin [1], since they do not place any condition on the cardinality of the separator. Note that if $\kappa_D(S, T) = |S|$, then S itself is the nearest (S, T)-separator, since $(D - S) \downarrow S = \emptyset$.

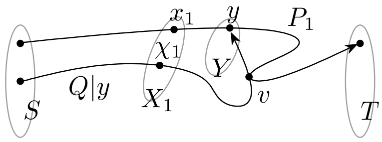

Fig. 1. The dipaths and (S, T)-separators in Lemma 2.2.

Intuitively, a nearest (S, T)-separator X with cardinality less than δ^+ forces 'many' arcs to be left in $D[(D - X) \downarrow S]$, and thus dicycles as well as 'many' (S, X)-dipaths may be found. This should lend some justification as to why we find them useful in attacking the Hoàng–Reed Conjecture — every place a separator with few vertices occurs, a number of dicycles must form above that separator.

In the following lemmas on nearest separators, if p_i and p_j are vertices of a dipath $P = p_1 p_2 \cdots p_a$, then we say p_j is *preceded* by p_i (or, p_i *precedes* p_j) on P, if $i \leq j$.

Lemma 2.2. *Let X be a nearest (S, T)-separator. Let $\{P_i\}_{i=1}^k$ be a set of $k = \kappa_D(S, T)$ disjoint (S, T)-dipaths. The following implication now holds. If y is a vertex of any (S, T)-separator, then it appears on P_j for some j, and is preceded by a vertex of X on P_j.*

Proof. The statement is trivial if $k = 0$, or $X = S$, since X and $(D - X) \downarrow S$ are empty, respectively. Suppose $|S| > k \geq 1$. Let W be the collection of all (S, T)-separators. Then, for each $Y \in W$ and dipath P_i, there is exactly one vertex of Y on P_i. Let X' consist of the first vertex, x_i, of P_i that lies in $\bigcup W$ for each $i = 1, \ldots, k$. We will show that X' is an (S, T)-separator. Suppose to the contrary that there is an (S, T)-dipath Q in $D - X'$. Let y be the first vertex of Q that lies on some (S, T)-separator $Y \in W$. Without loss of generality, y lies on P_1. Let v be the first vertex of Q to lie on $y|P_1$ (we might have $v = y$). If we follow Q and then $v|P_1$, we have an (S, T)-dipath $Q' = Q|v \cup v|P_1$ in $D - X'$, since y is preceded by x_1 on P_1. By definition of X', there is an $X_1 \in W$ such that $x_1 \in X_1$. Since X_1 is an (S, T)-separator, there is a vertex $\chi_1 \in X_1$ that lies on Q'. Since $\chi_1 \neq x_1$, the vertex χ_1 does not lie on P_1 and therefore χ_1 lies in $Q|v - v$. However, this contradicts the choice of y as the first vertex of $\bigcup W$ to lie on Q. See Fig. 1.

We will now show that X' is also an (S, X)-separator. Clearly, $|X'| = \kappa_D(S, X)$, so its cardinality is correct. Suppose to the contrary that there is an (S, X)-dipath Q in $D - X'$. Let x be the terminal vertex of Q. Without loss of generality, x lies on P_1. Let v be the first vertex of Q to lie on $x|P_1$. By the definition of X', the vertex $x_1 \in X'$ precedes x on P_1. Therefore, $Q|v \cup v|P_1$ is an (S, T)-dipath in $D - X'$, which is a contradiction. Since X' is an (S, X)-separator, there is no vertex of X in $(D - X') \downarrow S$. It is now easy to see that $(D - X') \downarrow S \subset (D - X) \downarrow S$ and so $X' = X$, by the definition of X as a nearest (S, T)-separator. □

If two (S, T)-separators X and Y both satisfy the conclusion of Lemma 2.2, then they must be identical. To see this, observe that $x_i \in X \cap V(P_i)$ and $y_i \in Y \cap V(P_i)$ both precede each other on P_i. Therefore, a nearest (S, T)-separator is unique. This also shows that the converse of Lemma 2.2 is trivial: if any (S, T)-separator X satisfies the conclusion of Lemma 2.2, then it is equal to the nearest (S, T)-separator.

Lemma 2.3. *Let Y be an (S, T)-separator. Then, X is the nearest (S, T)-separator if and only if X is the nearest (S, Y)-separator.*

Proof. Let k and $\{P_i\}_{i=1}^k$ be defined as in the previous lemma. We may also assume that $|S| > k \geq 1$, as in the previous lemma.

($\Longrightarrow$) Let X be the nearest (S, T)-separator. If X is not an (S, Y)-separator, then there is an (S, Y)-dipath Q in $D - X$. Let y be the terminal vertex of Q. Without loss of generality, y lies in P_1. Let v be the first vertex of Q to lie on $y|P_1$ (we might have $v = y$). If we follow Q and then $v|P_1$, we have an (S, T)-dipath $Q|v \cup v|P_1$ in $D - X$, since the vertex of $X \cap V(P_1)$ precedes y on P_1, by Lemma 2.2. This is impossible, so X is an (S, Y)-separator.

To see that X is a nearest (S, Y)-separator, let Y' be the nearest (S, Y)-separator and for each $i = 1, \ldots, k$, let y_i be the vertex of $Y \cap V(P_i)$. We will show that X satisfies the conclusion of Lemma 2.2 for the dipaths $\{P_i|y_i\}_{i=1}^k$. Since Y' is an (S, Y)-separator and any (S, T)-dipath contains a vertex of Y, it is clear that Y' is also an (S, T)-separator. Thus, by Lemma 2.2, the vertex x_i of $X \cap V(P_i)$ precedes the vertex y'_i of $Y' \cup V(P_i)$ on P_i, for each $i = 1, \ldots, k$. It is now trivial that x_i also precedes y'_i on $P_i|y_i$.

($\Longleftarrow$) Let X be the nearest (S, Y)-separator. Let X' be the nearest (S, T)-separator. By the first part of this proof, X' is also the nearest (S, Y)-separator. By Lemma 2.2, $X = X'$ since nearest separators are unique. □

Lemma 2.4. *Let X be the nearest (S, T)-separator. If $u \in (D - X) \downarrow S$, then $\kappa_D(S, X \cup \{u\}) = |X| + 1$.*

Proof. Suppose to the contrary that $\kappa_D(S, X \cup \{u\}) \leq |X|$. In this case we must have $\kappa_D(S, X \cup \{u\}) = |X|$, since there are $|X|$ disjoint (S, X)-dipaths. Let Y be an $(S, X \cup \{u\})$-separator. Then, Y is also an (S, T)-separator since any (S, T)-dipath contains a vertex of X. It is easy to see that any vertex of $(D - Y) \downarrow S$ is also a vertex of $(D - X) \downarrow S$, since any (S, v)-dipath in $D - Y$ contains no vertices of X. Since $u \in ((D - X) \downarrow S) \setminus ((D - Y) \downarrow S)$, this contradicts the minimality of $(D - X) \downarrow S$. □

Before moving on, we will only briefly remark that the collection of lower sets $(D - X) \downarrow S$, where X is an (S, T)-separator, can be shown to form a lattice (ordered by set containment). Here, the lower set corresponding to the nearest (S, T)-separator is the minimum point in the lattice.

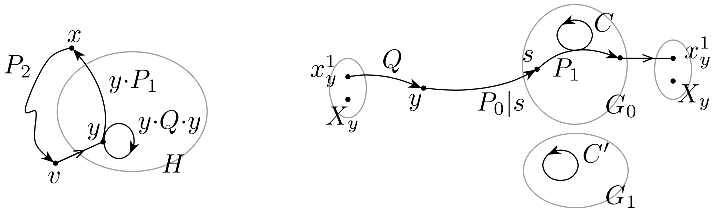

Fig. 2. The 2-circuit-forest in Proposition 3.1 and the 3-circuit-forest in Claim 1.

3. Hoàng–Reed for $\delta^+ = 2$

The following proposition introduces a basic idea behind the proofs of the Main Lemma and the Main Theorem.

Proposition 3.1. *Let D be a strongly connected digraph with $\delta^+(D) \geq 2$. Let $(y, x) \in V(D) \times V(D), y \neq x$, be such that $(D - x) \downarrow y$ is minimal with respect to inclusion. If $N_D^-(y) \setminus ((D - x) \downarrow y) \neq \emptyset$, then there are two dicycles C_1 and C_2 such that $V(C_1) \cap V(C_2) = \{y\}$.*

Proof. Let $H = D[(D - x) \downarrow y]$. Choosing H as such gives it three useful properties: (1) H is non-trivial due to the minimum out-degree of D; (2) H is a strong component of $D - x$, for otherwise there is a vertex w of a minimal strong component H' of H and $A_H^+(V(H')) = \emptyset$ means that $(D - x) \downarrow w = V(H')$ is a proper subset of $V(H)$ (which is impossible due to the minimality of $(D - x) \downarrow y$); and (3) x is a nearest $(N_D^+(y), x)$-separator in D, for otherwise there is a nearest $(N_D^+(y), x)$-separator, say $x' \neq x$, and $(D - x') \downarrow y$ is a proper subset of $V(H)$.

By Lemma 2.4 with $u \in N_H^-(y)$, there are two disjoint $(N_D^+(y), \{u, x\})$-dipaths P_1 and Q in D. Without loss of generality, the terminal vertex of P_1 is x. Let $v \in N_D^-(y) \setminus V(H)$ and let P_2 be an (x, v)-dipath in D. Then, set $C_1 = y \cdot P_1 \cup P_2 \cdot y$ and $C_2 = y \cdot Q \cdot y$ to obtain the conclusion. See Fig. 2. □

This proposition easily implies the $\delta^+ = 2$ case of the Hoàng–Reed Conjecture: first, if D is not strongly connected, then we simply restrict our attention to a minimal strong component D' of D, which also satisfies the minimal out-degree conditions, since $A_D^+(V(D')) = \emptyset$. Now, assuming that D is strongly connected, find a pair (y', x) such that $(D - x) \downarrow y'$ is minimal. By (2) in the above proof, the induced subdigraph $D[(D-x) \downarrow y']$ is strongly connected, so $(D-x) \downarrow y = (D-x) \downarrow y'$ for all $y \in (D - x) \downarrow y'$. Since D is strongly connected, there is some $y \in (D - x) \downarrow y'$ with $N_D^-(y) \setminus ((D - x) \downarrow y) \neq \emptyset$. Proposition 3.1 now applies to (y, x), and so there are two dicycles that intersect precisely at y.

To prepare for the next section, on $\delta^+ = 3$, we now prove the Main Lemma.

Main Lemma. *Let D be a digraph. Let s and t be two distinct vertices of D such that there is a (v, t)-dipath for all $v \in V(D)$. If $d_D^+(v) \geq 2$ for all $v \in V(D) \setminus \{t\}$, then D contains an (s, t)-dipath P and a dicycle C such that $|V(P) \cap V(C)| = 1$ and $t \notin V(C)$.*

Proof. We carry out induction on the number of vertices of D. Since $s \neq t$ and $d_D^+(s) \geq 2$, there are at least three vertices. If D has three vertices, then $D - t$ is a dicycle C of length 2, and each vertex of C dominates t. Let $P = st$ to obtain the conclusion.

Suppose the statement holds for all instances with fewer vertices. If $D - t$ is not strongly connected, let R be a shortest dipath in $D - t$ from s to a minimal strong component of $D - t$. Let s' be the terminal vertex of R. Let $D' = D[(D - A_D^+(t)) \downarrow s']$. Then, the vertices of some maximal strong component of D are not present in D', but $t \in D' \downarrow v$ and $d_{D'}^+(v) \geq 2$ for all $v \in V(D') \setminus \{t\}$. By induction, there is an (s', t)-dipath $R' \subset D'$ and a dicycle $C \subset D' - t$ such that $\left|V(R') \cap V(C)\right| = 1$. Let $P = R \cup R'$ to obtain the conclusion.

Now, assume that $D - t$ is strongly connected. Let t' be a nearest $(N_D^+(s), t)$-separator. If $t' \neq t$, then set $D' = D[(D - A_D^+(t')) \downarrow s]$. Then, $t \notin V(D')$, but $t' \in D' \downarrow v$ and $d_{D'}^+(v) \geq 2$ for all $v \in V(D') \setminus \{t'\}$. By induction, there is an (s, t')-dipath $R' \subset D'$ and a dicycle $C \subset D' - t'$ such that $\left|V(R') \cap V(C)\right| = 1$. Let R be an (t', t)-dipath and set $P = R' \cup R$ to obtain the conclusion. If $t' = t$, then by Lemma 2.4 with $u \in N_{D-t}^-(s)$, there are disjoint $(N_D^+(s), \{t, u\})$-dipaths Q_1 and Q_2. Without loss of generality, the terminal vertex of Q_1 is t. Let $P = s \cdot Q_1$ and $C = s \cdot Q_2 \cdot s$ to obtain the conclusion. □

With an application of the Main Lemma we can see that the 2-circuit-forests of the $\delta^+ = 2$ case of the Hoàng–Reed Conjecture can be chosen with a little more specification, as in the following proposition.

Proposition 3.2. *Let D be strongly connected digraph with $\delta^+(D) \geq 2$. For each arc $ts \in A(D)$, there are dicycles C and C' such that $ts \in A(C')$, $t \notin V(C)$ and $\left|V(C) \cap V(C')\right| = 1$.*

Proof. Apply the Main Lemma to get an (s, t)-dipath P and a dicycle C, such that $|V(P) \cap V(C)| = 1$ and $t \notin V(C)$. Let $C' = P \cdot s$ to obtain the 2-circuit-forest $C \cup C'$. □

Note that all of the proofs in this section just rely on Lemma 2.4, which is really just a straightforward consequence of Menger's Theorem.

4. Hoàng–Reed for $\delta^+ = 3$

As was mentioned in the Introduction, the Hoàng–Reed Conjecture trivially holds for δ^+-strongly connected digraphs. For digraphs that are not δ^+-strongly connected, nearest separators provide a useful way to organize the connectivity structure. Roughly, our attack on the $\delta^+ = 3$ case of the Hoàng–Reed Conjecture relies on sorting out what happens in the digraphs that are given to exist by Thomassen's Theorem 1.3.

We start with the following lemma that places some restrictions on a minimum counterexample to the Main Theorem.

Lemma 4.1. *Let D be a digraph with $\delta^+(D) \geq 3$ such that D does not contain a 3-circuit-forest. If D has minimum order and minimum size, then $\Delta^+(D) = 3$, $\delta^-(D) \geq 2$, and D is 2-strongly connected.*

Proof. If D has $\Delta^+(D) \geq 4$, then delete extra out-arcs at each vertex creating D' with $\Delta^+(D') = \delta^+(D') = 3$. Then, $D' \subset D$ and it contains a 3-circuit-forest, since $\left|A(D')\right| < |A(D)|$. Also, we can see that D must be strongly connected since, otherwise, a minimal strong component H of D is a proper subdigraph with $\delta^+(H) \geq 3$ and it contains a 3-circuit-forest.

The minimum in-degree of D is at least 1, since D is strongly connected. Suppose to the contrary that v is a vertex with $d_D^-(v) = 1$. Say, $uv \in A(D)$. Suppose that there is a vertex $w \in N_D^+(v)$ such that $w \notin N_D^+(u)$ and $w \neq u$. Consider the digraph D' with the vertices $V(D) \setminus \{v\}$ and arcs $A(D - v) \cup \{uw\}$. Since $\delta^+(D') \geq 3$ and D' has fewer vertices than D, there is a 3-circuit-forest F' in D'. We may assume that $uw \in A(F')$ for otherwise $F' \subset D$. Let F be the digraph with the vertices $V(F') \cup \{v\}$ and arcs $A(F - uw) \cup \{uv, vw\}$. Then, F is a 3-circuit-forest contained in D. Now suppose that no such w exists. This implies that $N_D^+(v) \subset N_D^+(u) \cup \{u\}$ and so uvu is a dicycle of D. Since $\delta^+(D - v) \geq 2$, by Proposition 3.2 $D - v$ contains a 2-circuit-forest F. Then, $F \cup uvu$ is a 3-circuit-forest in D.

To see that D is 2-strongly connected, suppose to the contrary that $D - x$ is not strongly connected for some $x \in V(D)$. Then, there are two distinct strong components G_0 and G_1 of $D - x$, where G_0 is a minimal strong component and G_1 is a maximal strong component. Since $\delta^+(D) \geq 3$ we have $\min\{d_{D-x}^+(v) : v \in V(G_0)\} \geq 2$, and since $\delta^-(D) \geq 2$ we have $\min\{d_{D-x}^-(v) : v \in V(G_1)\} \geq 1$. No arcs leave G_0 in $G - x$ since it is a minimal strong component, so $\delta^+(G_0) \geq 2$. By Proposition 3.2, G_0 contains a 2-circuit-forest F. Similarly, $\delta^-(G_1) \geq 1$ and so G_1 contains a dicycle C. The 3-circuit-forest $C \cup F$ is contained in D and so $D - x$ must be strongly connected for all $x \in V(D)$. □

In the proof of the Main Theorem we start with a minimal counterexample D and observe that it is 2-strongly connected, but not 3-strongly connected. In particular, according to the paragraph preceding Theorem 1.3, the out-neighborhood of any vertex is not 3-connected to the in-neighborhood. By using the fact that D must avoid any 3-circuit-forests, and using nearest separators (which all have two vertices) to disconnect in-neighborhoods from out-neighborhoods, we will partition D into a collection of acyclic subdigraphs. The next lemma says that no such partition can exist.

First, recall the following definitions. If $B \subset V(D)$, then the in-neighborhood of B is $N_D^-(B) = \bigcup_{u \in B} N_D^-(u) \setminus B$. An *acyclic* digraph is one which contains no dicycles. And $\mathcal{B} = \{B_1, B_2 \ldots, B_k\}$ is a *partition* of $V(D)$ if $\emptyset \neq B_i \subset V(D)$ for each i; $B_i \cap B_j = \emptyset$ for $i \neq j$; and each vertex of $V(D)$ is contained in some B_i.

Lemma 4.2. *Let D be a digraph with $\delta^+(D) = \Delta^+(D) = 3$. Then, there is no partition $\mathcal{B}$ of $V(D)$ such that, for each $B \in \mathcal{B}$, the induced subdigraph $D[B]$ is acyclic and $\left|N_D^-(B)\right| = 2$.*

Proof. Suppose to the contrary that we have such a partition $\mathcal{B}$. For each $y \in V(D)$, let B_y denote the set in $\mathcal{B}$ such that $y \in B_y$. For convenience, say $N_y = N_D^-(B_y)$ for each $y \in V(D)$. First, we will show that

$$\begin{cases} \left|A_D^+(B_y)\right| \geq 3, \quad \text{and } \left|A_D^-(B_y)\right| \leq 2, \quad \text{if } \left|B_y\right| = 1; \\ \left|A_D^+(B_y)\right| \geq 5, \quad \text{and } \left|A_D^-(B_y)\right| \leq 4, \quad \text{if } \left|B_y\right| = 2; \\ \left|A_D^+(B_y)\right| \geq 6, \quad \text{and } \left|A_D^-(B_y)\right| \leq 6, \quad \text{if } \left|B_y\right| \geq 3. \end{cases} \tag{$*$}$$

Since $\left|N_y\right| = 2$, we have $\left|A_D^-(B_y)\right| \leq 2\left|B_y\right|$. Since $\Delta^+(D) = 3$, and $A_D^-(B_y) \subset A_D^+(N_y)$ we have $\left|A_D^-(B_y)\right| \leq 3\left|N_y\right| = 6$. This establishes the upper bounds for $\left|A_D^-(B_y)\right|$.

To bound $\left|A_D^+(B_y)\right|$, note that a minimal vertex v_0 (a trivial minimal strong component) of $D[B_y]$ has out-degree zero in $D[B_y]$, since $D[B_y]$ is acyclic. There is at least one minimal strong component, so $\delta^+(D) = 3$ means $\left|A_D^+(B_y)\right| \geq 3$. If B_y has at least two vertices, then a minimal vertex v_1 of $D[B_y] - v_0$ has out-degree at most 1 in $D[B_y]$, and so $\left|A_D^+(B_y)\right| \geq 3 + 2 = 5$. Similarly, if B_y has at least three vertices, then a minimal vertex v_2 of $D[B_y] - v_0 - v_1$ has out-degree at most 2 in $D[B_y]$, and so $\left|A_D^+(B_y)\right| \geq 3 + 2 + 1 = 6$. Note that $\left|A_D^+(B_y)\right| = 6$ only if $D[\{v_0, v_1, v_2\}]$ is a transitive tournament and $A_D^+(\{v_0, v_1, v_2\}) = A_D^+(B_y)$. This establishes the lower bounds.

Our next goal is to show that $\left|B_y\right| \geq 3$, and $\left|A_D^+(B_y)\right| = \left|A_D^-(B_y)\right| = 6$ for all $y \in V(D)$. For $i = 1, 2$, let $\mathcal{B}_i$ be the collection of B_y with $\left|B_y\right| = i$. Let $\mathcal{B}_3$ be the collection of B_y with $\left|B_y\right| \geq 3$. Clearly, $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 = \mathcal{B}$ is a partition of $V(D)$ and we have $\sum_{B \in \mathcal{B}} \left|A_D^+(B)\right| = \sum_{B \in \mathcal{B}} \left|A_D^-(B)\right|$. By $(*)$, we have

$$\sum_{B \in \mathcal{B}} \left|A_D^+(B)\right| = \sum_{B \in \mathcal{B}_1} \left|A_D^+(B)\right| + \sum_{B \in \mathcal{B}_2} \left|A_D^+(B)\right| + \sum_{B \in \mathcal{B}_3} \left|A_D^+(B)\right| \geq 3\left|\mathcal{B}_1\right| + 5\left|\mathcal{B}_2\right| + 6\left|\mathcal{B}_3\right|$$

and similarly, $\sum_{B\in\mathcal{B}} \left|A_D^-(B)\right| \le 2|\mathcal{B}_1| + 4|\mathcal{B}_2| + 6|\mathcal{B}_3|$. Putting these together, we have

$$3|\mathcal{B}_1| + 5|\mathcal{B}_2| + 6|\mathcal{B}_3| \le 2|\mathcal{B}_1| + 4|\mathcal{B}_2| + 6|\mathcal{B}_3|$$

which implies $|\mathcal{B}_1| = |\mathcal{B}_2| = 0$. Moreover, $\sum_{B\in\mathcal{B}} \left|A_D^-(B)\right| \le 6|\mathcal{B}| \le \sum_{B\in\mathcal{B}} \left|A_D^+(B)\right|$ and equality holds only if $\left|A_D^+(B)\right| = \left|A_D^-(B)\right| = 6$ for each $B \in \mathcal{B}$.

We are now in position to complete this proof. For each $y \in V(D)$, let T_y be the set of vertices incident with tails of the arcs of $A_D^+(B_y)$. As was noted in the argument proving $(*)$, $\left|A_D^+(B_y)\right| = 6$ only if T_y has exactly three vertices and they induce a transitive tournament, $D[T_y]$. Consider a fixed $y \in V(D)$ and let u be the vertex of $D[T_y]$ with in-degree zero. Since u dominates exactly two vertices of T_y, there is a unique vertex v of $N_D^+(u) \setminus B_y$. Therefore, $B_v \ne B_y$ and uv is an arc of $A_D^-(B_v)$. This means that $u \in N_v$ and u contributes exactly one arc to $A_D^-(B_v)$. Since N_v contains exactly two vertices and $\Delta^+(D) = 3$, there are at most four arcs in $A_D^-(B_v)$. This is a contradiction since we proved above that $\left|A_D^-(B)\right| = 6$ for each $B \in \mathcal{B}$. □

The preceding lemma can be interpreted as generalization of the simple observation that, in a digraph with minimum out-degree at least 3, there must be some vertex with in-degree at least 3.

Main Theorem. *If D is a digraph with $\delta^+(D) \ge 3$, then D contains a 3-circuit-forest.*

Proof. Let D be a minimum counterexample according to Lemma 4.1. If D is 3-strongly connected, then for any vertex $u \in V(D)$, the out-neighborhood is 3-connected to the in-neighborhood so there are three disjoint $(N_D^+(u), N_D^-(u))$-dipaths $\{P_i\}_{i=1}^3$, and $\bigcup_{i=1}^3 u \cdot P_i \cdot u$ is a 3-circuit-forest. This is impossible, so for each vertex $y \in V(D)$, there is an $(N_D^+(y), N_D^-(y))$-separator $X_y = \{x_y^1, x_y^2\}$. Choose X_y to be the nearest $(N_D^+(y), N_D^-(y))$-separator. Since the out-degree of each vertex is 3, a minimal strong component of $D - X_y$ is non-trivial. Let H_y be the vertex set of a minimal strong component of $D - X_y$. Let $B_y = V(D - X_y) \setminus H_y$. We will argue with y as an arbitrary vertex of D, that is, each claim will hold for all $y \in V(D)$. Note that even though X_y is unique by Lemma 2.2, H_y and B_y might not be uniquely defined.

Claim 1 establishes the uniqueness of H_y. Claims 2, 3 and 4 establish some 'local' properties of X_y, H_y and B_y for fixed y. Finally, Claim 5 looks at all of the sets $\{B_y : y \in V(D)\}$ together and establishes how they relate globally.

Claim 1. *$D[H_y]$ is the only minimal strong component of $D - X_y$.*

Let $G_0, \ldots, G_m$ be the minimal strong components of $D - X_y$. Suppose $m > 0$. Let P_0 be a (y, X_y)-dipath of D. Without loss of generality, P_0 passes through G_0 and no other G_i. Let s be the first vertex of P_0 in G_0. Let $t \notin V(D)$ be a new vertex and let G_0' be the digraph with the vertices $V(G_0) \cup \{t\}$ and the arcs $A(G_0) \cup \{gt : gx \in A(D), g \in V(G_0), x \in X_y\}$. Then, by the Main Lemma applied to G_0', there is an (s, t)-dipath $P \subset G_0'$ and a dicycle $C \subset G_0' - t$ such that $|V(P) \cap V(C)| = 1$. Let P_1 be the dipath $P - t$. Without loss of generality, the terminal vertex of P_1 dominates $x_y^1 \in X_y$. Let Q be an (x_y^1, y)-dipath in $D - x_y^2$. Then, the dicycles $P_0|s \cup P_1 \cdot Q$ and C form a 2-circuit-forest F. By adding a dicycle C' of G_1, we obtain a 3-circuit-forest $C' \cup F$. See Fig. 2. □

Note that since H_y is uniquely defined, so too is B_y. The next claim is analogous to part of the hypothesis of Proposition 3.1.

Claim 2. *H_y is minimal with respect to inclusion among all sets $(D - X') \downarrow \chi$, where $D - X'$ is not strongly connected, $\left|X'\right| = 2$ and $\chi \in V(D) \setminus X'$.*

Suppose to the contrary that $H' = (D - X') \downarrow \chi$ is a proper subset of H_y. We may take $(D - X') \downarrow \chi$ to be minimal with respect to inclusion. Note that this implies that $D[H']$ is strongly connected. There is a vertex $x' \in X' \setminus X_y$, for otherwise $X' = X_y$ and $H' = H_y$. By Lemma 2.4 with $u = x'$, there are three disjoint $(N_D^+(y), X_y \cup \{x'\})$-dipaths $\{P_i\}_{i=1}^3$. We may assume that the terminal vertex of P_3 is x' and the terminal vertex of P_i is x_y^i for $i = 1, 2$. Since D is 2-strongly connected, there are two disjoint $(X_y, N_D^-(y))$-dipaths Q_1, Q_2, where the initial vertex of Q_i is x_y^i. The dipath Q_i intersects with P_j only on X_y for $i, j \in \{1, 2\}$, since otherwise if they intersect at the vertex $v \notin X_y$, then $P_j|v \cup v|Q_i$ is a walk from $N_D^+(y)$ to $N_D^-(y)$ that does not intersect X_y. If P_1 and P_2 do not intersect H', then the two dicycles $y \cdot P_1 \cup Q_1 \cdot y$ and $y \cdot P_2 \cup Q_2 \cdot y$ form a 2-circuit-forest F. By adding a dicycle C of $D[H']$, we obtain a 3-circuit-forest $C \cup F$.

We may therefore assume that P_2 intersects H'. Since H' is not 3-connected to X_y, the dipath P_1 is disjoint with H'. Let x'' be the vertex of X' contained in P_2. Note that P_1 and $x''|P_2$ are disjoint dipaths that terminate in X_y. Let s be the first vertex of P_2 in H'. Note that X' must be a nearest $(N_D^+(s), X_y)$-separator, since otherwise there is a $(N_D^+(s), X_y)$-separator $Y \ne X'$ such that $(D - Y) \downarrow s$ is a proper subset of H', which contradicts the choice of H'. By Lemma 2.4 with $u \in N_D^-(s) \cap H'$ (u exists since $D[H']$ is strongly connected), there are three disjoint $(N_D^+(s), X' \cup \{u\})$-dipaths $\{R_i\}_{i=1}^3$. We may assume that R_1 terminates at u and R_2 terminates at x''. Then, the two dicycles $y \cdot P_1 \cup Q_1 \cdot y$ and $y \cdot P_2|s \cdot R_2 \cup x''|P_2 \cup Q_2 \cdot y$ form a 2-circuit-forest F. The dicycle $C = s \cdot R_1 \cdot s$ intersects F precisely at s, so $C \cup F$ is a 3-circuit-forest. See Fig. 3. □

Now we use Claim 2 to show that we can find dipaths and dicycles in $D[H_y]$ that are analogous to those found in Proposition 3.1.

Claim 3. *For each vertex w of H_y, there are two (w, X_y)-dipaths $Q_1, Q_2 \subset G[H_y \cup X_y]$ and a dicycle $C' \subset G[H_y]$ that pairwise intersect on exactly w.*

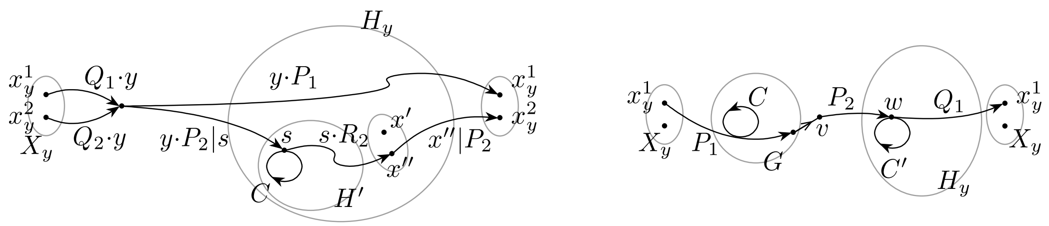

Fig. 3. The 3-circuit-forests in Claims 2 and 4, respectively.

By Claim 2, X_y is a nearest $(N_D^+(w), X_y)$-separator, since otherwise there is an $(N_D^+(w), X_y)$-separator, say $X' \neq X$, such that $(D - X') \downarrow w$ is a proper subset of H_y. By Lemma 2.4 with $u \in N_D^-(w) \cap H_y$ (u exists since $D[H_y]$ is strongly connected), there are three disjoint $(N_D^+(w), X_y \cup \{u\})$-dipaths $\{P_i\}_{i=1}^3$. We may assume that the terminal vertex of P_3 is u. Then, set $Q_i = w \cdot P_i$ for $i = 1, 2$ and $C' = w \cdot Q_3 \cdot w$ to obtain the conclusion. □

Claim 3 shows that $D[H_y]$ contains many useful ingredients for a 3-circuit-forest. Claim 4 shows that, in order to avoid 3-circuit-forests in D, the rest of the digraph must be acyclic.

Claim 4. *The induced subdigraph $D[B_y \cup X_y]$ is acyclic.*

Suppose to the contrary that $D[B_y \cup X_y]$ contains a non-trivial strong component G. Then, each (X_y, H_y)-dipath in D must pass through G. If this is not the case, let P_1 be an (X_y, H_y)-dipath that avoids G. Let w be the terminal vertex of P_1 (it is the only vertex of P_1 in H_y). By Claim 3 we obtain the dipaths Q_1, Q_2 and the dicycle C'. We may assume that the terminal vertex of Q_1 is the initial vertex of P_1. Then, the dicycles $P_1 \cup Q_1$ and C' form a 2-circuit-forest F. By adding a dicycle C in G, we obtain a 3-circuit-forest $C \cup F$.

We may therefore assume that each (X_y, H_y)-dipath in D passes through G. Let $t \notin V(D)$ be a new vertex and let G' be the digraph with the vertices $V = ((D - A_D^+(V(G))) \downarrow X_y) \cup \{t\}$ and the arcs $A(D[V]) \cup \{gt : gv \in A(D), g \in V(G), v \notin V(G)\}$. Notice that $d_{G'}^+(u) \geq 2$ for all $u \in V(G') \setminus \{t\}$. By the Main Lemma applied to G' with $s = x_y^1$, there is an (x_y^1, t)-dipath $P \subset G'$ and a dicycle $C \subset G' - t$ such that $|V(C) \cap V(P)| = 1$. Let P_1 be the dipath $P - t$. Let $v \notin V(G)$ be a vertex dominated by the terminal vertex of P_1. Let P_2 be a (v, H_y)-dipath. Let w be the terminal vertex of P_2. By Claim 3 we obtain the dipaths Q_1, Q_2 and the dicycle C'. We may assume that the terminal vertex of Q_1 is x_y^1. Then, the dicycles C, C' and $P_1 \cdot P_2 \cup Q_1$ form a 3-circuit-forest. See Fig. 3. □

Since y was arbitrary, Claim 4 has shown that $D[B_y]$ is acyclic for each $y \in V(D)$. Now we will consider how the sets $\{B_y : y \in V(D)\}$ interrelate. Note that, for two distinct vertices $u \neq v$, the sets B_u and B_v may be identical. The next claim shows that either $B_u = B_v$ or they are disjoint.

Claim 5. *The collection of sets $\mathcal{B} = \{B_y : y \in V(D)\}$ forms a partition of $V(D)$.*

Since $D[H_y]$ is strongly connected and y lies on no dicycle of $D - X_y$, it is clear that $y \in B_y$ for each $y \in V(D)$. So $\mathcal{B}$ covers all vertices of D. Now we will show that if $y \in B_u$ for some vertex $u \in V(D)$, then $B_y = B_u$. By Claim 4, $D[B_u \cup X_u]$ is acyclic, so there is no $(N_D^+(y), N_D^-(y))$-dipath in $D - X_u$. This shows that X_u is an $(N_D^+(y), N_D^-(y))$-separator. By definition, X_y is a nearest $(N_D^+(y), N_D^-(y))$-separator. Therefore, Lemma 2.3 says that X_y is also a nearest $(N_D^+(y), X_u)$-separator.

Since $D[B_u]$ is acyclic and $\delta^+(D) \geq 3$, there are three disjoint $(N_D^+(y), H_u \cup X_u)$-dipaths in D because, otherwise, a separator X' with two vertices would cause a minimum strong component of $D[B_u] - X'$ to contain a dicycle. Therefore, in $D - X_y$, there is either an $(N_D^+(y), X_u)$-dipath or an $(N_D^+(y), H_u)$-dipath, say P. The first possibility cannot occur, since X_y is an $(N_D^+(y), X_u)$-separator. So P exists and we can label its terminal vertex as w. By Claim 2, H_u is minimal with respect to inclusion, so either $(D - X_y) \downarrow w$ contains a vertex of X_u or $(D - X_y) \downarrow w = (D - X_u) \downarrow w$. For the first possibility, if Q is a (w, X_u)-dipath in $D - X_y$, then $P \cup Q$ is an $(N_D^+(y), X_u)$-dipath in $D - X_y$, which again cannot occur. The second possibility implies that $X_y = X_u$, so by Claim 1 we have $H_y = H_u$ and also $B_y = B_u$, which proves the claim. □

At this point, we have a partition $\mathcal{B}$ of $V(D)$ such that $D[B]$ is acyclic for each $B \in \mathcal{B}$, but to apply Lemma 4.2, we need to consider $\left|N_D^-(B)\right|$. Since $D[H_y]$ is a minimal strong component of $D - X_y$, the head of each arc of $A_D^+(H_y)$ is in X_y. The sets B_y and H_y partition $V(D - X_y)$, so $A_D^-(B_y) \subset A_D^+(X_y)$. Further, since X_y is 2-connected to B_y, both vertices of X_y are incident with tails of $A_D^-(B_y)$. Therefore, $N_D^-(B_y) = X_y$ and, in particular, it consists of two vertices. By Lemma 4.2, no such partition can exist so we have a contradiction, and the theorem is proved. □

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